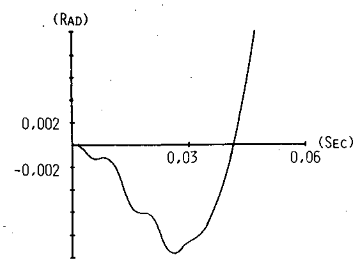

Fig. 5 Bending angle at m_2, unstable controller.

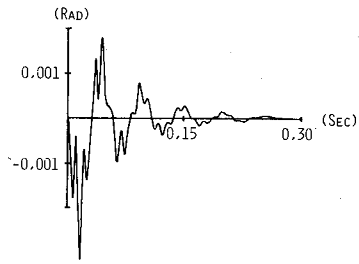

Fig. 6 Bending angle at m_2, optimized controller.

gyroscopic effects assigned to the elastic part of the system can destabilize the whole motion. On the other hand, positive restoring terms result in asymptotic stable behavior.

Application

The developed procedure is now applied to derive the equations of motion of the Large Space Telescope, which is a nonrotating flexible structure (Fig. 2). Since $\boldsymbol{\Omega}=\mathbf{0}$, C_2, and C_3 vanish, and for a plane motion in z direction, C_l reduces to a vector expression $\boldsymbol{c}$

$$c=\int_0^l \left\{\frac{y}{l}\bar{w}+\frac{l}{m}\rho I_x\frac{\partial\bar{w}}{\partial y}\right\}\mathrm{d}y+\sum_{i=1}^{3}\left\{\frac{m_i}{m}y\bar{w}+\frac{l}{m}J_{xi}\frac{\partial\bar{w}}{\partial y}\right\}\Big|_{y_i},\qquad (m=m_{el}) \tag{23}$$

The eigenfunctions $\bar{w}_i$ are plotted in Fig. 3; they are derived using 12 eigenfunctions of a free-free beam as approximation functions. Introducing modal damping, the equations of motion read

$$\ddot{x}+\mathrm{diag}(2\zeta\omega_i)\dot{x}+\mathrm{diag}(\omega_i^2)x=-c\ddot{\alpha}+\frac{\partial\bar{w}}{\partial y}\Big|_{y_M}M_\alpha+\bar{w}\,|_{y_F}F_D \tag{24}$$

$$(J/m)\ddot{\alpha}=-c^T\ddot{x}+M_\alpha+y_F F_D \tag{25}$$

where F_D is a disturbance force at $y=y_F$ and M_α a control torque at y_M. It is assumed, that the system is excited with an eccentric shock at $y_F=3/4\ l$. Figure 4 shows the uncontrolled system response (bending angle at m_2). If a control torque is considered which is time varying with a frequency near a bending frequency, the whole motion becomes unstable (Fig. 5). However, if M_α is optimized with respect to a quadratic cost functional using the well-known optimal regulator theory, then the rigid body motion as well as the bending oscillation can be kept small (Fig. 6).

References

[1] Bremer, H.,"Dynamics and Control of Flexible Hybrid Structures," available from National Technical Information Service, Springfield, Va. 22151, as N78-25456.

[2] Likins, P.W. and Fleischer, E.G., "Large-Deformation Modal Coordinates for Nonrigid Vehicle Dynamics," Jet Propulsion Laboratory, Pasadena, Calif., JPL Tech. Rept. 32-1565, 1972, p. 20.

[3] Dym, C.L. and Shames, I.H., *Solid Mechanics,* McGraw Hill, New York, 1973.

[4] Collatz, L., "Eigenwertaufgaben mit technischen Anwendungen," Akademische Verlagsgesellschaft, Leipzip, 1963.

[5] Meirovitch, L., *Analytical Methods in Vibrations,* Macmillan, New York, 1967.

Eigensolution for Large Flexible Spacecraft with Singular Gyroscopic Matrices

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Introduction

THE advent of the large space structures (LSS) presents new problems in the dynamics and control areas. The large sizes envisioned and practical limits on lifting capacity require that the structural members be extremely light, which in turn implies a large degree of distributed structural flexibility and very low natural frequencies. No longer can the effects of the spacecraft flexibility be treated as a perturbation of the rigid body. In determining the system response to external excitation or to internal disturbances, the problem of mathematical modeling becomes critical, particularly if one wishes to obtain the response by modal analysis. The theory for the modal analysis of nonspinning systems with elastic restoring forces is well developed. The presence of stored angular momentum in nonspinning spacecraft, however, complicates the response problem because of the resulting gyroscopic effects. What complicates the response problem for a linear gyroscopic system is that the classical modal analysis will not uncouple the system equations of motion, so that the question of spacecraft modes requires a different interpretation. The question has been studied extensively by Meirovitch[1] for systems with a nonsingular gyroscopic matrix. A method was developed for determining the spacecraft modes, as well as a modal analysis for the evaluation of the response of gyroscopic systems.[2] This Note uses the free-free classical modes of the spacecraft (with all rotors locked) to reformulate the equations of motion using a reduced state vector. An extension must be made to Meirovitch's method[1] because of the singularity of the gyroscopic matrix involved in the present formulation. Subsequently, the problem is transformed into an eigenvalue problem for a single symmetric matrix.

Problem Formulation

Spinning rotors are employed as momentum exchange controllers to stabilize the spacecraft attitude, appearing in the form of reaction wheels, momentum wheels, or control moment gyros. Any combination of these can be present in a spacecraft and they can be distributed over the spacecraft structure. Our interest lies in constant relative speed of the rotor and constant relative orientation of its spin axis with respect to the mounting location. The spacecraft structure is assumed to be stabilized relative to an inertial space, so that

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the spacecraft itself constitutes a gyroscopic system. The free vibration problem is represented by the homogeneous system

$$m\ddot{q}+g\dot{q}+kq=0 \tag{1}$$

where m and k are symmetric matrices, g is a skew symmetric singular matrix and q is the configuration vector. The matrix k is singular because the structure is unrestrained. The classical normal modes are the solution of the eigenvalue problem

$$m\phi\Lambda^2=k\phi \tag{2}$$

where ϕ is the modal matrix and Λ^2 is the diagonal matrix of the eigenvalues. Note that Eq. (2) represents the eigenvalue problem obtained from that associated with Eq. (1) by letting $g=0$. Hence, it is the eigenvalue problem of the system with the rotors locked (where ϕ is the modal matrix for this system and Λ is the diagonal matrix of the associated natural frequencies). For controllers of small mass and inertia, the classical modes will not be greatly affected by relocation of the locked rotors. We shall assume that the modes are normalized so as to satisfy

$$\phi^Tm\phi=U \qquad \phi^Tk\phi=\Lambda^2 \tag{3}$$

where U is the identity matrix.

The matrix ϕ is not the modal matrix for the system described by Eq. (1), however, so that it will not uncouple the system. Indeed, introducing the linear transformation,

$$q=\phi\eta \tag{4}$$

into Eq. (1) and multiplying by ϕ^T we can write

$$\ddot{\eta}+\phi^Tg\phi\dot{\eta}+\Lambda^2\eta=0 \tag{5}$$

where $\phi^Tg\phi$ is a skew symmetric matrix, by definition nondiagonal. It is also singular.

The matrix Λ^2 contains six zero diagonal elements corresponding to the zero frequencies associated with the rigid body modes. It will prove convenient to partition Λ^2 and η as follows:

$$\Lambda^2=\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \Lambda_E^2 \end{array}\right] \qquad \eta=\left[\begin{array}{c} \eta_R \\ \hline \eta_E \end{array}\right] \tag{6}$$

where the subscripts R and E refer to rigid and elastic modes, respectively. Clearly, if the system is of order n, the Λ_E^2 is an $(n-6)\times(n-6)$ matrix, η_R is a six-dimensional vector, and η_E is an $(n-6)$-dimensional vector. With the introduction of the identity

$$\Lambda_E^2\dot{\eta}_E-\Lambda_E^2\dot{\eta}_E=0 \tag{7}$$

the set of second-order differential equations, Eq. (5), can be transformed into the set of $2n-6$ first-order equations

$$I\dot{x}+Gx=0 \tag{8}$$

where x is the "reduced" state vector

$$x=\begin{Bmatrix} \dot{\eta} \\ \eta_E \end{Bmatrix} \tag{9}$$

$$I=\left[\begin{array}{c|c} U & 0 \\ \hline 0 & \Lambda_E^2 \end{array}\right] \qquad G=\left[\begin{array}{c|c} \phi^Tg\phi & \begin{array}{c} 0 \\ \hline \Lambda_E^2 \end{array} \\ \hline 0 \;|\; -\Lambda_E^2 & 0 \end{array}\right] \tag{10}$$

Clearly, the matrix I is a positive definite diagonal matrix. On the other hand, G is a skew symmetric singular matrix.

Solution Technique

Because I is diagonal (and positive definite), the equations of motion can be regarded as having a skew symmetric form, so that the associated eigenvalue problem yields complex eigenvalues and eigenvectors. The problem described by Eq. (8) is similar to that treated by Meirovitch[1] with the exception that here the matrix G is singular.

It will prove of interest to review briefly the problem treated in Ref. 1. The eigenvalue problem had the form

$$\lambda Ix+Gx=0 \tag{11}$$

where I is symmetric and positive definite, and G is skew symmetric and nonsingular. It is known that the solution to the eigenvalue problem (11) consists of n pairs of pure imaginary complex conjugates $\lambda_r=\pm i\omega_r$, and n pairs of complex conjugate eigenvectors x_r and $\bar{x}_r$ $(r=1,2,\ldots,n)$. To obtain a solution of the eigenvalue problem, it is actually more convenient to work with real quantities instead of complex ones. Therefore, let us insert $\lambda_r=i\omega_r$ and $x_r=y_r+iz_r$ into Eq. (11), where y_r is the real part and z_r the imaginary part of the vector x_r. Then, separate the real and imaginary parts of the equation to obtain

$$\begin{aligned} -\omega_rIz_r+Gy_r&=0 \\ \omega_rIy_r+Gz_r&=0 \end{aligned} \qquad (r=1,2,\ldots,n) \tag{12}$$

This can be reduced to

$$\begin{aligned} \omega_r^2Iy_r&=Ky_r \\ \omega_r^2Iz_r&=Kz_r \end{aligned} \qquad (r=1,2,\ldots,n) \tag{13}$$

where

$$K=G^TI^{-1}G \tag{14}$$

is a real symmetric matrix, $K=K^T$, because I is a real symmetric matrix and G is a real skew symmetric matrix. The solution of this reformulated eigenvalue problem consists of n pairs of repeated eigenvalues ω_r^2 (ω_r^2 has double multiplicity) and n pairs of associate eigenvectors y_r and z_r $(r=1,2,\ldots,n)$.

This Note extends this method to the case in which G is singular. This is done in a formal way by analyzing the null spaces of G and K and showing them to be equal. Thus, the restriction that G be nonsingular is removed, and the method can be used to obtain a solution of Eq. (8).

In order to prove that the null spaces are equal, first assume that x is a member of the null space of G

$$Gx=0 \tag{15}$$

It is then obvious that x is also in the null space of K

$$Kx=G^TI^{-1}Gx=0 \tag{16}$$

Now, let us assume that x is in the null space of K. Then we may set the inner product to zero

$$(G^TI^{-1}Gx,\,x)=0 \tag{17}$$

If we rewrite K as the product of $I^{-1/2}G$ and its transpose, then we may take advantage of the adjoint relationship

$$(G^TI^{-1}Gx,\,x)=-(I^{-1/2}Gx,\,I^{-1/2}Gx)=0 \tag{18}$$

Because Eq. (18) represents the norm of a vector, and because I is positive definite, $I^{-\frac{1}{2}}Gx$ itself must be zero. Therefore, x is shown to be in the null space of G. Moreover, because the null spaces of G and K are equal, the multiplicity of the zero eigenvalue is unchanged by the reformulation. This was not the case for the nonzero eigenvalues where the reformation, Eq. (13), doubled the multiplicity.

After the solution of the reformulated eigenvalue problem has been obtained, the rigid-body mode component of the eigenvector has to be reconstructed from the velocity information, as it is not explicitly present in the reduced state vector. The velocity term is written

$$\dot{\eta}=\lambda\eta \tag{19}$$

where $\lambda=i\omega$. Denoting by η_1 and η_2 the real and imaginary part of the modal vector η, respectively, the velocity becomes

$$\dot{\eta}=-\omega\eta_2+i\omega\eta_1 \tag{20}$$

Then, the real and imaginary parts of the eigenvector x_r are simply

$$y_r=\begin{Bmatrix} -\omega\eta_2 \\ \eta_{1_E} \end{Bmatrix}_r, \quad z_r=\begin{Bmatrix} \omega\eta_1 \\ \eta_{2_E} \end{Bmatrix}_r \tag{21}$$

In order to construct the eigenvector q_r in the configuration space, we recall Eq. (4) and write

$$q_r=\phi\eta_r=\phi\eta_{1r}+i\phi\eta_{2r} \tag{22}$$

where η_{1r} and η_{2r} are the real and imaginary parts of the eigenvector η_r.

Conclusions

A practical method is presented that allows the analyst to properly assess the effect of stored angular momentum on the system vibration properties. From a numerical standpoint, the method is attractive because it uses existing capabilities to model the structure with the rotors locked, and then avoids complex quantities by executing an eigenvalue analysis of a reformulated symmetric matrix. This method leads directly to the solution of the response problem.[2] Parametric analysis of the system for different rotor angular momentum requires the execution of the reformulated gyroscopic eigenvalue problem only. Advantage may be taken of the ability to truncate the number of input modes in order to make the parametric analysis less burdensome. The reformulated gyroscopic problem may also be run alone to study the effect of different rotor locations. This assumes that the rotors are small so that when their locations are changed, they do not greatly affect the classical modes.

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References

[1] Meirovitch, L., "A New Method of Solution of the Eigenvalue Problem for Gyroscopic Systems," *AIAA Journal*, Vol. 12, Oct. 1974, pp. 1337-1342.

[2] Meirovitch, L., "A Modal Analysis for the Response of Linear Gyroscopic Systems," *Journal of Applied Mechanics*, Vol. 42, Feb. 1975, pp. 446-450.

Stationkeeping at Constant Distance

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Introduction

A STATIONKEEPING procedure is described in which a vehicle circles about an orbiting body while maintaining constant separation distance. The origin of the problem was a requirement for the Orbiter of the Space Shuttle to inspect and check out a deployed payload. Most previous studies of relative motion of bodies close to one another have focused on the terminal phase of rendezvous,[1-4] while others have considered flight in formation without the requirement to view all sides of one of the vehicles.[5-7] One study which presents examples of the circling type of relative orbits[8] does not consider the problem of maintaining a constant separation distance between the two bodies, which is of essential interest in the current study.

Stationkeeping at Constant Distance

The Clohessy-Wiltshire equations[1] describe the relative motion of two bodies in orbit. These equations were examined and a solution found that results in the Orbiter making a circular orbit relative to the payload so that the separation distance between the Orbiter and payload remains constant at 3000 ft, a value chosen from safety considerations. The equations in the coordinate system employed are:

$$x=2(2\dot{x}_0/\omega-3z_0)\sin\omega t-(2\dot{z}_0/\omega)\cos\omega t$$
$$+(6\omega z_0-3\dot{x}_0)t+(x_0+2\dot{z}_0/\omega)$$

$$y=y_0\cos\omega t+(\dot{y}_0/\omega)\sin\omega t$$

$$z=(2\dot{x}_0/\omega-3z_0)\cos\omega t+(\dot{z}_0/\omega)\sin\omega t+(4z_0-2\dot{x}_0/\omega)$$

where t is time, s; x is the distance along reference orbit plus downrange, ft; y is the distance normal to reference orbit, ft (makes right-hand system with x and z); z is the distance below reference orbit, ft; ω is 2π/orbit period, 1/s; and subscript 0 denotes initial condition.

Note that the motion in the crossrange (y) direction is independent of the vertical and downrange z and x motion. The amplitude of the periodic motion in the downrange direction can be seen to be twice that in the vertical direction.

For the purposes of this analysis, the following assumptions are made concerning the initial conditions.

At $t=0$, assume that 1) a phasing orbit has been completed so that the payload will be in the center of the stationkeeping orbit, $x_0=-(2\dot{z}_0/\omega)$, 2) otherwise, the Shuttle orbit is the same as the payload orbit, $y_0=z_0=0$, and 3) to prevent drift, $\dot{x}_0=0$.

Now, consider motion in the x-z plane only. This is the vertical plane aligned along the direction of motion in the parking orbit.

$$x=-(2\dot{z}_0/\omega)\cos\omega t$$

$$z=(\dot{z}_0/\omega)\sin\omega t$$

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